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THE IMPACT OF SWAPPING RISKS IN FACILITATING CAPITAL INVESTMENT IN E-COMMERCE

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ABSTRACT

A swapping scheme is proposed to facilitate capital flow into e-commerce by controlling credit risks associated with e-commerce corporations. More specifically, we develop and analyze a mathematical model for swapping credit risks across two industrial sectors: industrial sector A, without involving e-commerce; and industrial sector B, which relies upon e-commerce. When two Banks X and Y provide loans to corporations in A and B, a swapping scheme can be devised between Banks X and Y to improve the Value-at-Risk for both of them. By exploiting the dynamic stochastic model based on a Markov Modulated Poisson Process (MMPP) developed by Takada, Sumita and Takahashi[1], efficient computational procedures are established for solving the Value-at-Risk problems.

Keywords: Uniformization Procedure, Laplace Transform, Convolution

1. INTRODUCTION

Businesses that use e-commerce are the core of the new and growing internet-based economy. However, the collapse of the IT industry in 2002, combined with the Lehman shock in 2008, caused the capital flow into e-commerce to slow significantly, so it is important to find a way to facilitate capital flow into e-commerce by controlling risks involved. The purpose of this paper is to develop and analyze a mathematical model for swapping credit risks across two industrial sectors: industrial sector A, without involving e-commerce and industrial sector B, which relies upon e-commerce. When two banks X and Y provide loans to corporations in A and B, it is shown that a swapping scheme can be devised between X and Y that improves the Value-at-Risk for both X and Y, thereby providing incentive for X and Y to increase their involvement with corporations in e-commerce. More specifically, we consider two banks, X and Y, which provide loans to corporations in two industrial sectors: A, which is not involved in e-commerce, and B, which is involved in e-commerce. Let $C_{X:A}$ be a group of corporations in A to which the bank X provides loans, and define $C_{X:B}$, $C_{Y:A}$ and $C_{Y:B}$ similarly. It is assumed that $C_A = C_{X:A} \cap C_{Y:A} \neq \phi$ and $C_B = C_{X:B} \cap C_{Y:B} \neq \phi$. Let $|S|$ be the cardinality of a set S. Then the number of corporations can be written as $M_A = |C_A|$ and $M_B = |C_B|$.

The two banks *X* and *Y* agree to sign a swapping contract in the following manner. Let *K* be a portion of the loans provided to corporations in $C_A \cup C_B$ by each bank. This amount *K* common to *X* and *Y* is subject to the swapping contract. However, the risks for *X* to bear against *A* and *B* may be different from those for *Y*. We assume that the potential risk of each corporation in C_A is identical to *X* and that the potential risk of each corporation in C_B is also the same as *X*. Similar assumptions are taken for *Y*. Let $\overline{L}^{X:A}$ be the amount of loans that *X* provides to each corporation in C_A, and define $\overline{L}^{X:B}$, $\overline{L}^{Y:A}$ and $\overline{L}^{Y:B}$ similarly. Then it is clear that

$$\overline{L}^{X:A} M_A + \overline{L}^{X:B} M_B = K\,;\, \overline{L}^{Y:A} \cdot M_A + \overline{L}^{Y:B} \cdot M_B = K \tag{1.1}$$

For computational simplicity, we normalize the equations (1.1) by dividing both sides of each equation by *K*. Let $L^{I:J} = \overline{L}^{I:J} / K$ so that

$$L^{X:A} M_A + L^{X:B} M_B = 1;\, L^{Y:A} M_A + L^{Y:B} M_B = 1 \tag{1.2}$$

Let $D_A(t)$ and $D_B(t)$ be the number of defaults by time *t* in C_A and the number of defaults by time *t* in C_B, respectively. Assuming that a default of a corporation results in complete loss of the entire loan for both *X* and *Y*, the underlying credit risks for *X* and *Y* can be written as

$$\ell_X(L^{X:A}, L^{X:B}, t) = L^{X:A} D_A(t) + L^{X:B} D_B(t) \tag{1.3}$$

$$\ell_Y(L^{Y:A}, L^{Y:B}, t) = L^{Y:A} D_A(t) + L^{Y:B} D_B(t) \tag{1.4}$$

In a swapping scheme in which each bank takes $100 \times p$ percent of the credit risk of the other bank, the credit risks of *X* and *Y* with this swapping can be written as:

$$\ell_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t) = (1-p) \cdot \ell_X(L^{X:A}, L^{X:B}, t) + p \cdot \ell_Y(L^{Y:A}, L^{Y:B}, t) \tag{1.5}$$

$$\ell_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t) = p \cdot \ell_X(L^{X:A}, L^{X:B}, t) + (1-p) \cdot \ell_Y(L^{Y:A}, L^{Y:B}, t) \tag{1.6}$$

To see how this swapping scheme could improve the credit risks of both *X* and *Y*, we introduce the Value-at-Risks as:

$$VaR_X(L^{X:A}, L^{X:B}, t, \alpha) = \inf\left\{\beta \mid \Pr\left\{\ell_X(L^{X:A}, L^{X:B}, t) \geq \beta\right\} \leq \alpha\right\} \tag{1.7}$$

$$\begin{aligned} &VaR_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha) \\ &\quad = \inf\left\{\beta \mid \Pr\left\{\ell_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t) \geq \beta\right\} \leq \alpha\right\} \end{aligned} \tag{1.8}$$

$$VaR_Y(L^{Y:A}, L^{Y:B}, t, \alpha) = \inf\left\{\beta \mid \Pr\left\{\ell_Y(L^{Y:A}, L^{Y:B}, t) \geq \beta\right\} \leq \alpha\right\} \tag{1.9}$$

$$\begin{aligned} &VaR_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha) \\ &\quad = \inf\left\{\beta \mid \Pr\left\{\ell_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t) \geq \beta\right\} \leq \alpha\right\} \end{aligned} \tag{1.10}$$

If the distributions of $D_A(t)$ and $D_B(t)$ are obtained, those of $\ell_X(L^{X:A}, L^{X:B}, t)$ and $\ell_Y(L^{Y:A}, L^{Y:B}, t)$ can be evaluated based on (1.3) and (1.4), enabling one to calculate the distributions of $\ell_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t)$ and $\ell_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t)$ from (1.5) and (1.6). Finally, the Value-at-Risks in (1.7) - (1.10) can be assessed accordingly.

To evaluate the distributions of $D_A(t)$ and $D_B(t)$, we rely on the MMPP (Markov Modulated Poisson Process) approach originally proposed by Takada, Sumita and Takahashi[1] to cope with multiple industrial sectors. Because of the rapidly changing market environments in e-commerce, we adopt this approach to model differences between the default process in *A*, which does not involve e-commerce, and that in *B*, which does not rely on e-commerce.

The structure of this paper is as follows. In Section 2, the underlying default processes in *A* and *B* are introduced. Section 3 provides a succinct summary of the computational algorithms for evaluating the joint distribution of $[J(t), D_A(t), D_B(t)]$ based on[1]. Section 4 provides numerical examples, given $L^{X:A}$, $L^{X:B}$, $L^{Y:A}$ and $L^{Y:B}$, the distribution functions of $\ell_X(L^{X:A}, L^{X:B}, t)$ and $\ell_Y(L^{Y:A}, L^{Y:B}, t)$ are plotted for different values of *t*, and we demonstrate that the swapping contract could benefit both banks *X* and *Y* by choosing the swapping rate *p* appropriately. Some concluding remarks are given in Section 5.

Throughout the paper, matrices and vectors are indicated by the double underscore and the single underscore respectively (e.g., $\underline{\underline{a}}$, $\underline{m}$). A submatrix of $\underline{\underline{a}}$ restricted to row indices in *G* and column indices *B* is written as $\underline{\underline{a}}_{GB} = [a_{ij}]_{i \in G, j \in B}$. Given a matrix $\underline{\underline{a}} = [a_{ij}]$, the *i*-th row sum is denoted by a_i, and the associated diagonal matrix $\underline{\underline{a}}_D$ is defined as $\underline{\underline{a}}_D = diag\{a_i\}, \quad a_i = \sum_j a_{ij}$.

2. MODEL DESCRIPTION

We consider a credit risk model with two industrial sectors *A* and *B*, with M_A and M_B being active corporations at time $t=0$. Defaults of corporations are influenced by changes in macro economic conditions that affect the default intensities of the two industrial sectors differently. More formally, let $\{J(t): t \geq 0\}$ be a Birth-Death process on $J = \{0,1,\ldots,J\}$, describing the macro economic condition at time *t*, governed by upward transition rates ν_i^+, $(i=0,1,\ldots,J-1)$ and downward transition rates ν_i^-, $(i=1,\ldots,J)$. When $J(t) = i \in J$, individual active corporations in *A* and *B* have the default intensities $\xi_A(i)$ and $\xi_B(i)$, respectively.

Given $[i, \underline{m}] = [i, m_A, m_B]$, let the integrated industrial default intensities for *A* and *B* be denoted by $\lambda_A(i, \underline{m})$ and $\lambda_B(i, \underline{m})$, respectively. Since there are m_A and m_B active corporations in *A* and *B*,

$$\lambda_A(i, \underline{m}) = m_A \times \xi_A(i) \,;\, \lambda_B(i, \underline{m}) = m_B \times \xi_B(i) \qquad (2.1)$$

In addition, at the time of a state change of *J(t)*, at most one active corporation in *A* or *B* may default instantaneously with certain probability, but never both at a once. We denote the probability of having no defaults upon occurrence of a transition of *J(t)* from *i* to *j* by Θ_{ij}. A default occurs with probability $1-\Theta_{ij}$; let Θ_{ij}^{A} and Θ_{ij}^{B} represent the probability of this default's occurring in *A* and the probability of its occurring in *B*, respectively. Accordingly, $0 \le \Theta_{ij}, \Theta_{ij}^{A}, \Theta_{ij}^{B} \le 1$ and $\Theta_{ij} + \Theta_{ij}^{A} + \Theta_{ij}^{B} = 1$.

3. COMPUTATIONAL ALGOROTHMS

This section provides a succinct summary of the computational algorithms for evaluating the joint distribution of $[J(t), D_A(t), D_B(t)]$ based on [1]. The trivariate process can be formulated as a sophisticated bivariate MMPP with respect to two industrial sectors *A* and *B*, which bivariate is both temporally and spatially inhomogeneous. Based on the uniformization procedure of Keilson [2] combined with the dynamic first passage time analysis, the computational algorithms can be developed to evaluate the joint distribution of $[J(t), D_A(t), D_B(t)]$ so one can assess the joint distribution of the cumulative losses that are due to the defaults in *A* and in *B* by time *t*.

In order to capture the dynamic behavior of the bivariate MMPP, we introduce a fundamental absorbing trivariate process $[J(t), \underline{M}_{\underline{m}}(t)]$, given $\underline{m} = (m_A, m_B) > \underline{0}$, where the corresponding state space can be written as

$$S_{\underline{m}} = J \times M_{\underline{m}} = \{(i,\underline{n}) : i \in J, \underline{n} = (m_A, m_B), (m_A - 1, m_B) or (m_A, m_B - 1)\} \tag{3.1}$$

The fundamental absorbing trivariate process is constructed in such a way that its stochastic behavior is identical to that of $[J(t), \underline{M}(t)]$ if $\underline{M}_{\underline{m}}(t) = \underline{m}$, while all other states are made to be absorbing. The transition structure of $[J(t), \underline{M}_{\underline{m}}(t)]$ is depicted in Figure 3.1.

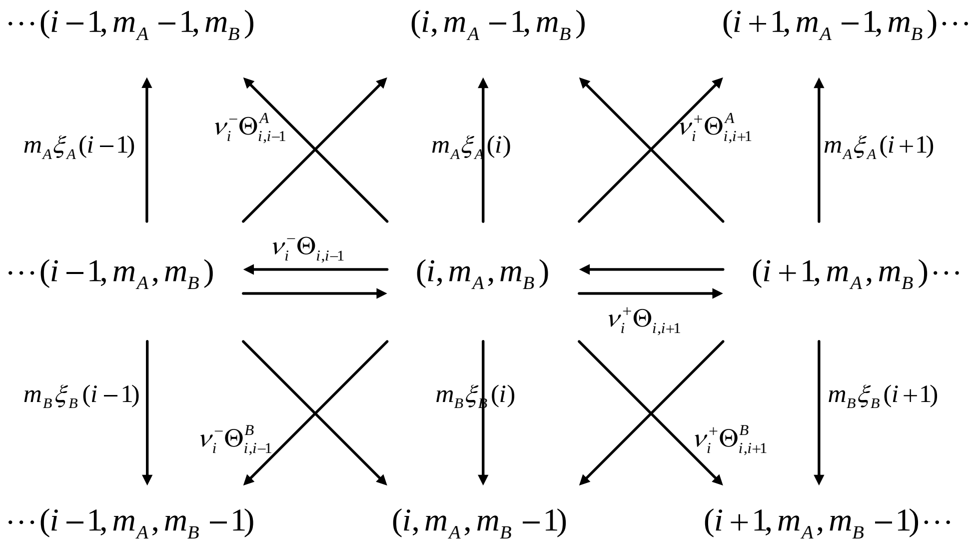

Figure 3.1. Transition Structure.

In order to define $[J(t), \underline{M}_{\underline{m}}(t)]$ in terms of hazard rate matrices, we decompose the state space $S_{\underline{m}}$ in (3.1) into three mutually exclusive and exhaustive subspaces:

$$G = \{(i, \underline{n}) : i \in J, \underline{n} = (m_A, m_B)\}$$
$$A = \{(i, \underline{n}) : i \in J, \underline{n} = (m_A - 1, m_B)\} \quad (3.2)$$
$$B = \{(i, \underline{n}) : i \in J, \underline{n} = (m_A, m_B - 1)\}$$

G corresponds to the set of states located in the first row of Figure 3.1, and the states in the second row and those in the third row of Figure 3.1 constitute *A* and *B*, respectively. With these definitions, let $\underline{\underline{\nu}}$ be the hazard rate matrix governing the transitions of $[J(t), \underline{M}_{\underline{m}}(t)]$ within *G*:

$$\underline{\underline{\nu}} = \begin{bmatrix} 0 & \nu_0^+\Theta_{0,1} & & & & \\ \nu_1^-\Theta_{1,0} & 0 & \nu_1^+\Theta_{1,2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \nu_i^-\Theta_{i,i-1} & 0 & \nu_i^+\Theta_{i,i+1} & \\ & & & \ddots & \ddots & \\ & & & & \nu_J^-\Theta_{J,J-1} & 0 \end{bmatrix} \quad (3.3)$$

$\underline{\underline{\nu}}$ is independent of $\underline{m}$, as indicated by its notation, provided that $0 < m_A < M_A$ and $0 < m_B < M_B$.

In order to capture the transitions of $[J(t), \underline{M}_{\underline{m}}(t)]$ from G to A and those from G to B, we define the two matrices $\underline{\underline{\Lambda}}_A(\underline{m})$ and $\underline{\underline{\Lambda}}_B(\underline{m})$ by

$$\underline{\underline{\Lambda}}_A(\underline{m}) = \begin{bmatrix} \lambda_A(0,\underline{m}) & \nu_0^+\Theta_{0,1}^A & & & \\ \nu_1^-\Theta_{1,0}^A & \lambda_A(1,\underline{m}) & \nu_1^+\Theta_{1,2}^A & & \\ & \ddots & \ddots & & \\ & & \nu_J^-\Theta_{J,J-1}^A & \lambda_A(J,\underline{m}) \end{bmatrix} \tag{3.4}$$

and

$$\underline{\underline{\Lambda}}_B(\underline{m}) = \begin{bmatrix} \lambda_B(0,\underline{m}) & \nu_0^+\Theta_{0,1}^B & & & \\ \nu_1^-\Theta_{1,0}^B & \lambda_B(1,\underline{m}) & \nu_1^+\Theta_{1,2}^B & & \\ & \ddots & \ddots & & \\ & & \nu_J^-\Theta_{J,J-1}^B & \lambda_B(J,\underline{m}) \end{bmatrix}. \tag{3.5}$$

The entire hazard rate matrix governing $[J(t), \underline{M}_{\underline{m}}(t)]$, denoted by $\underline{\underline{V}}(\underline{m})$, can then be written as:

$$\underline{\underline{V}}(\underline{m}) = \begin{bmatrix} \underline{\underline{\nu}} & \underline{\underline{\Lambda}}_A(\underline{m}) & \underline{\underline{\Lambda}}_B(\underline{m}) \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \tag{3.6}$$

The corresponding diagonal matrix is defined as:

$$\underline{\underline{V}}_D(\underline{m}) = \begin{bmatrix} \underline{\underline{\nu}}_D + \underline{\underline{\Lambda}}_{A:D}(\underline{m}) + \underline{\underline{\Lambda}}_{B:D}(\underline{m}) & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \tag{3.7}$$

To facilitate our analysis further, we introduce Keilson's [2] uniformization procedure. Given $\underline{m} \in M$ is a constant satisfying

$$\nu \geq \max_{[i,\underline{m}] \in J \times M} \{\nu_i + \lambda_A(i,\underline{m}) + \lambda_B(i,\underline{m})\}, \tag{3.8}$$

the stochastic matrix $\underline{\underline{a}}_\nu(\underline{m})$ associated with ν is then defined as

$$\underline{\underline{a}}_{\nu}(\underline{m}) = \underline{\underline{I}} - \tfrac{1}{\nu}\underline{\underline{\nu}}_{D}(\underline{m}) + \tfrac{1}{\nu}\underline{\underline{\nu}}(\underline{m}) \tag{3.9}$$

Let the transition probability matrix $\underline{\underline{P}}(\underline{m},t)$ be defined by

$$\underline{\underline{P}}(\underline{m},t) = \left\lfloor P_{(i,\underline{n})(i',\underline{n}')}(\underline{m},t) \right\rfloor_{(i,\underline{n}),(i',\underline{n}')\in S_{\underline{m}}}, \tag{3.10}$$

where

$$P_{(i,\underline{n})(i',\underline{n}')}(\underline{m},t) = \Pr\left\{J(t)=i', M_{\underline{m}(t)} = \underline{n}' \mid J(0)=i, M_{\underline{m}}(0)=\underline{n}\right\}. \tag{3.11}$$

The corresponding Laplace transform is defined by

$$\underline{\underline{\pi}}(\underline{m},s) = \int_0^t e^{-st}\,\underline{\underline{P}}(\underline{m},t)\,dt. \tag{3.12}$$

One then has

$$\underline{\underline{\pi}}_{GG}(\underline{m},s) = \frac{1}{s+\nu}\left[\sum_{k=0}^{\infty}\left(\tfrac{\nu}{s+\nu}\right)^k \underline{\underline{a}}_{\nu:GG}(\underline{m})^k\right], \tag{3.13}$$

which would play an important role.

If $T_{[i,\underline{n}]\to A\cup B}$ is the first passage time of $[J(t),\underline{M}_{\underline{m}}(t)]$ from $[i,\underline{n}]$ to $A\cup B$,

$$I(T_{[i,\underline{n}]\to A\cup B}) = \inf\left\{t : |\,[J(t),\underline{M}_{\underline{m}}(t)] \in A\cup B \mid [J(0),\underline{M}_{\underline{m}}(0)] = [i,\underline{n}]\right\}. \tag{3.14}$$

We introduce the following indicator

$$I(T_{[i,\underline{n}]\to A\cup B}) = \begin{cases} A & if\ \underline{M}_{\underline{m}}(T_{[i,\underline{n}]\to A\cup B}) \in A \\ B & if\ \underline{M}_{\underline{m}}(T_{[i,\underline{n}]\to A\cup B}) \in B \end{cases} \tag{3.15}$$

and define the joint distribution matrices of $T_{[i,\underline{n}]\to A\cup B}$ and $I(T_{[i,\underline{n}]\to A\cup B})$ as

$$\underline{\underline{S}}_{A}(\underline{m},t) = \left\lfloor \underline{\underline{S}}_{A:ij}(\underline{m},t) \right\rfloor;$$
$$\underline{\underline{S}}_{A:ij}(\underline{m},t) = \Pr\left\{T_{[i,\underline{n}]\to A\cup B} \le t, I(T_{[i,\underline{n}]\to A\cup B}) = A, J(t)=j \mid \underline{M}_{\underline{m}}(0)=\underline{n}, J(0)=i\right\} \tag{3.16}$$

and

$$\underline{\underline{S}}_{B}(\underline{m},t) = \left[\underline{\underline{S}}_{B:ij}(\underline{m},t)\right]$$
$$\underline{\underline{S}}_{B:ij}(\underline{m},t) = \Pr\left\{T_{[i,\underline{n}]\to A\cup B} \le t, I(T_{[i,\underline{n}]\to A\cup B}) = B, J(t) = j \mid \underline{M}_{\underline{m}}(0) = \underline{n}, J(0) = i\right\}$$
(3.17)

The corresponding Laplace transform matrices are denoted by

$$\underline{\underline{\sigma}}_{A}(\underline{m},s) = [\sigma_{A:ij}(\underline{m},s)];\ \sigma_{A:ij}(\underline{m},s) = \int_0^\infty e^{-st}\,dS_{A:ij}(\underline{m},t) \tag{3.18}$$

and

$$\underline{\underline{\sigma}}_{B}(\underline{m},s) = [\sigma_{B:ij}(\underline{m},s)];\ \sigma_{B:ij}(\underline{m},s) = \int_0^\infty e^{-st}\,dS_{B:ij}(\underline{m},t) \tag{3.19}$$

Through the uniformization procedure, Theorem 3.1 then holds true. (See Takada, Sumita and Takahashi [1].)

Theorem 3.1

$$a)\quad \underline{\underline{\sigma}}_{A}(\underline{m},s) = \sum_{k=0}^{\infty}\left(\tfrac{\nu}{s+\nu}\right)^{k+1}\underline{\underline{a}}_{\nu:GG}(\underline{m})^{k}\,\underline{\underline{a}}_{\nu:GA}(\underline{m});$$
$$\underline{\underline{\sigma}}_{B}(\underline{m},s) = \sum_{k=0}^{\infty}\left(\tfrac{\nu}{s+\nu}\right)^{k+1}\underline{\underline{a}}_{\nu:GG}(\underline{m})^{k}\,\underline{\underline{a}}_{\nu:GB}(\underline{m});$$
$$b)\quad \underline{\underline{s}}_{A}(\underline{m},\tau) = \sum_{k=0}^{\infty}e^{-\nu\tau}\,\tfrac{(\nu\tau)^k}{k!}\,\nu\underline{\underline{a}}_{\nu:GG}(\underline{m})^{k}\,\underline{\underline{a}}_{\nu:GA}(\underline{m});$$
$$\underline{\underline{s}}_{B}(\underline{m},s) = \sum_{k=0}^{\infty}e^{-\nu\tau}\,\tfrac{(\nu\tau)^k}{k!}\,\nu\underline{\underline{a}}_{\nu:GG}(\underline{m})^{k}\,\underline{\underline{a}}_{\nu:GB}(\underline{m});$$

Theorem 3.1 enables one to evaluate the joint distribution of $[J(t),\underline{M}_{\underline{m}}(t)]$ in the following manner: If $[J(t),\underline{M}_{\underline{m}}(t)] = [j,\underline{m}]$, starting with $[J(0),\underline{M}_{\underline{m}}(0)] = [i,\underline{M}]$, multiple paths connect $\underline{M}$ to $\underline{m}$, where each path represents a sequence of occurrences of defaults in *A* and *B*. The transition probability associated with each path can then be obtained by multiplying the matrices $\underline{\underline{\sigma}}_{A}(\underline{n},s)$ and $\underline{\underline{\sigma}}_{B}(\underline{n}',s)$ in such a way that the sequence of the path are reflected in the order of the matrix multiplications, along with choices of $\underline{n}$ and $\underline{n}'$, followed by the final matrix multiplication by $\underline{\underline{\pi}}_{GG}(\underline{m},s)$. More specifically, let the matrices $\underline{\underline{\Xi}}(m_A,m_B,s)$ be defined recursively for $m_A = M_A-1, M_A-2,\cdots,0$ and $m_B = M_B-1, M_B-2,\cdots,0$ as

$$\underline{\underline{\Xi}}(m_A, M_B, s) = \underline{\underline{\Xi}}(m_A + 1, M_B, s)\underline{\underline{\sigma}}_A(m_A + 1, M_B, s), \quad (3.20)$$
$$\underline{\underline{\Xi}}(m_A, m_B, s) = \underline{\underline{\Xi}}(m_A + 1, m_B, s)\underline{\underline{\sigma}}_A(m_A + 1, m_B, s)$$
$$+ \underline{\underline{\Xi}}(m_A, m_B + 1, s)\underline{\underline{\sigma}}_A(m_A, m_B + 1, s)$$
$$\underline{\underline{\Xi}}(M_A, m_B, s) = \underline{\underline{\Xi}}(M_A, m_B + 1, s)\underline{\underline{\sigma}}_A(M_A, m_B + 1, s),$$

starting with

$$\underline{\underline{\Xi}}(M_A, M_B, s) = \underline{\underline{I}}. \quad (3.21)$$

Then Theorem 3.2 holds true.

Theorem 3.2

For $m_A = M_A - 1, M_A - 2, \cdots, 0$ and $m_B = M_B - 1, M_B - 2, \cdots, 0$, if $\underline{\underline{\Xi}}(m_A, m_B, s)$ are defined in (3.20) and (3.21):

$$\underline{\underline{\beta}}(m_A, m_B, s \mid \underline{M}) = \underline{\underline{\Xi}}(m_A, m_B, s)\underline{\underline{\pi}}(m_A, m_B, s).$$

Equation (3.13), together with Theorems 3.1 and 3.2, enables one to compute the time-dependent joint probability of $[J(t), \underline{M}(t)]$, providing a computational vehicle for assessing the Value-at-Risk, as we show in section 4. See Takada, Sumita and Takahashi [1] for additional details.

4. NUMERICAL RESULTS

In this section, we numerically explore a swapping scheme between two banks that facilitate cash flow into e-commerce by applying the numerical algorithms described in Sections 3. For this purpose, we suppose that industrial sector *A* consists of a group of corporations that are not involved in e-commerce, while the corporations in industrial sector *B* rely upon e-commerce. We assume that industrial sector *A* has 40 corporations while industrial sector *B* contains 60 corporations.

We assume that the stochastic process *J(t)* describing the macro economic conditions follows the Ehrenfest process on $J=\{0,1,\ldots,2V\}$. Here, a higher state implies a better economic condition, with state *V* corresponding to a normal economic condition. The Ehrenfest process has the upward transition rate ν_i^+ and the downward transition rate ν_i^- at state $i \in J$, given by

$$\nu_i^+ = c \cdot (V - \tfrac{i}{2}); \nu_i^- = c \cdot \tfrac{i}{2} \quad (4.1)$$

where *c* is a positive constant.

The Ehrenfest process is chosen because of these dynamics; that is, as the process deviates from the normal economic condition, the driving force to bring the process back to the normal condition becomes stronger. (See e.g., [3].) Given the macro economic condition $i \in J$, the default intensities $\xi_A(i)$ and $\xi_B(i)$ are given by

$$\xi_A(i) = \alpha_A \cdot e^{-\beta_A(i-V)}; \xi_B(i) = \alpha_B \cdot e^{-\beta_B(i-V)} \tag{4.2}$$

Here, the macro economic condition is expressed as the Ehrenfest process characterized by (4.1), defined on $J=\{0,1,\ldots,2V\}$, with state 0 being the lowest, state $V=3$ being normal, and state $2V=6$ being the highest. The governing parameter c defined in (4.1) is given as $c=3.0$. Since the corporations in *B* (which is involved with e-commerce) are likely to be more fragile and responsive to the macro economic condition than the corporations in *A* (which is not), we set $\alpha_A = 0.01 < 0.04 = \alpha_B$ based on (4.2) and assume the exponential factors β_A and β_B are equal, with $\beta_A = \beta_B$ =0.3. For the same reason, the default probability upon transition of the macro economic condition for industrial sector *B* is higher than that for the industrial sector *A*, and then we set $\Theta^A_{i,j} = 0.048 < 0.078 = \Theta^B_{i,j}$.

Figures 4.1 and 4.2 depict the values of the Value-at-Risk problems for Banks *X* and *Y* without swapping; that is, $VaR_X(L^{X:A}, L^{X:B}, t, \alpha)$ and $VaR_Y(L^{Y:A}, L^{Y:B}, t, \alpha)$, given in (1.7) and (1.9), respectively, are plotted with $t=3$ and $\alpha = 0.01$. Similarly, the corresponding values with swapping are illustrated in Figures 4.3 and 4.4; that is, $VaR_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)$ and $VaR_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)$, as in (1.8) and (1.10), respectively, are plotted with $t=3$, $\alpha = 0.01$ and $p=0.16$.

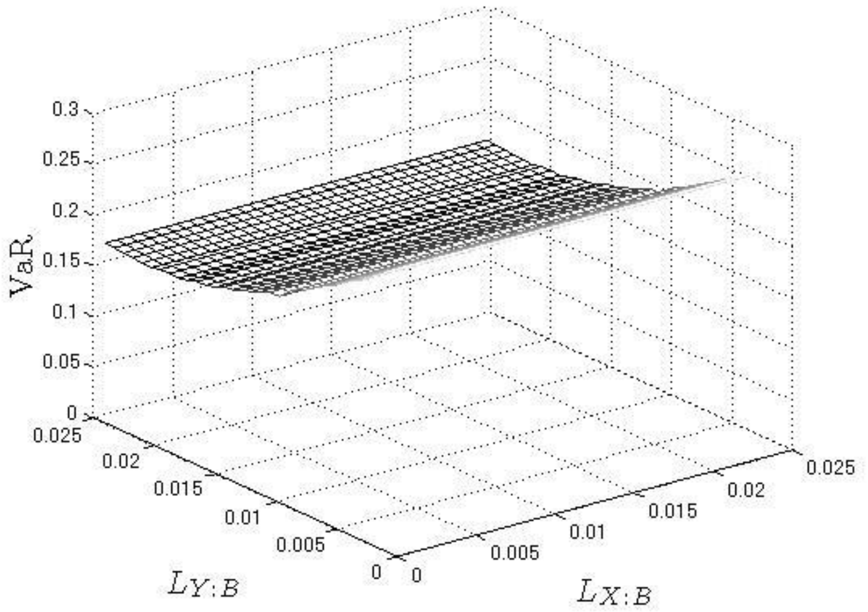

Figure 4.1. $VaR_X(L^{X:A}, L^{X:B}, t, \alpha)$

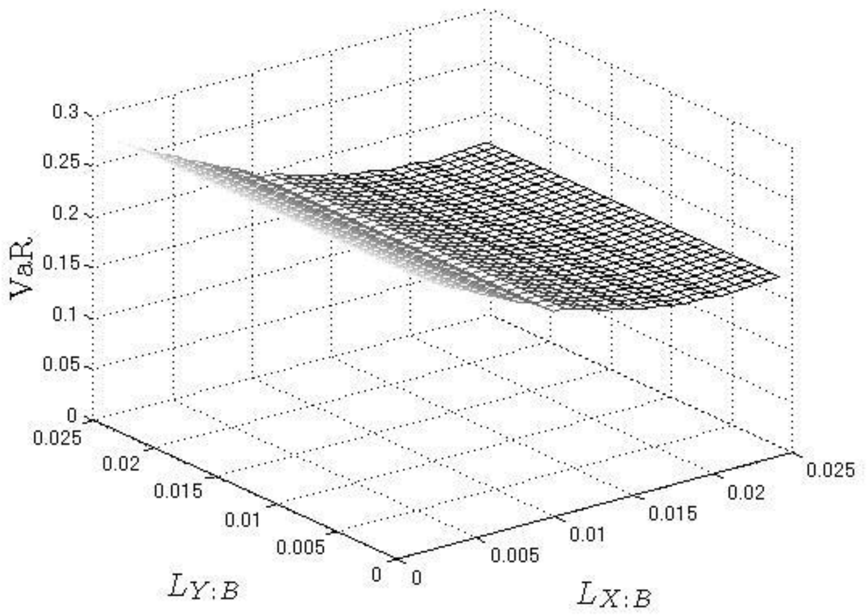

Figure 4.2. $VaR_Y(L^{Y:A}, L^{Y:B}, t, \alpha)$

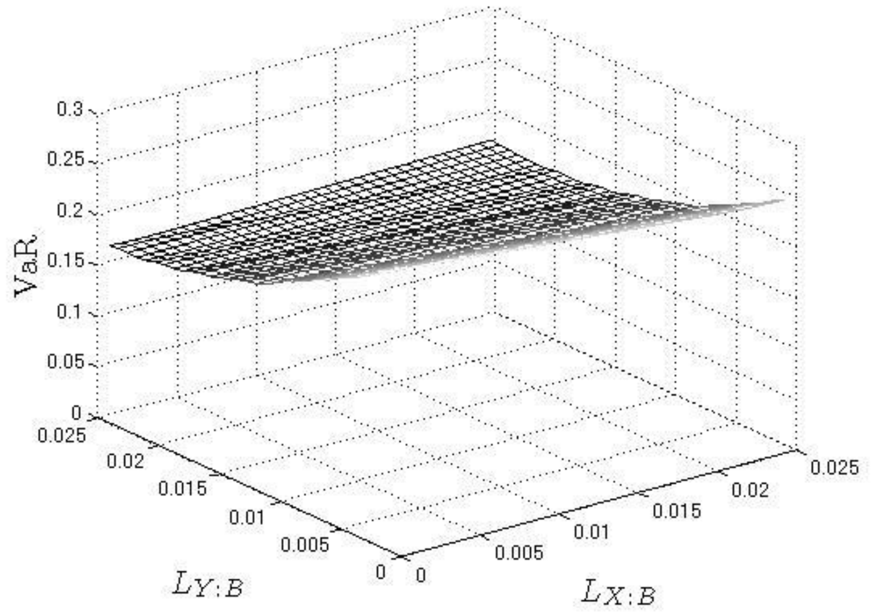

Figure 4.3. $VaR_{X:Swap}(\bullet, L^{X:B}, \bullet, L^{Y:B}, p, t, \alpha)$

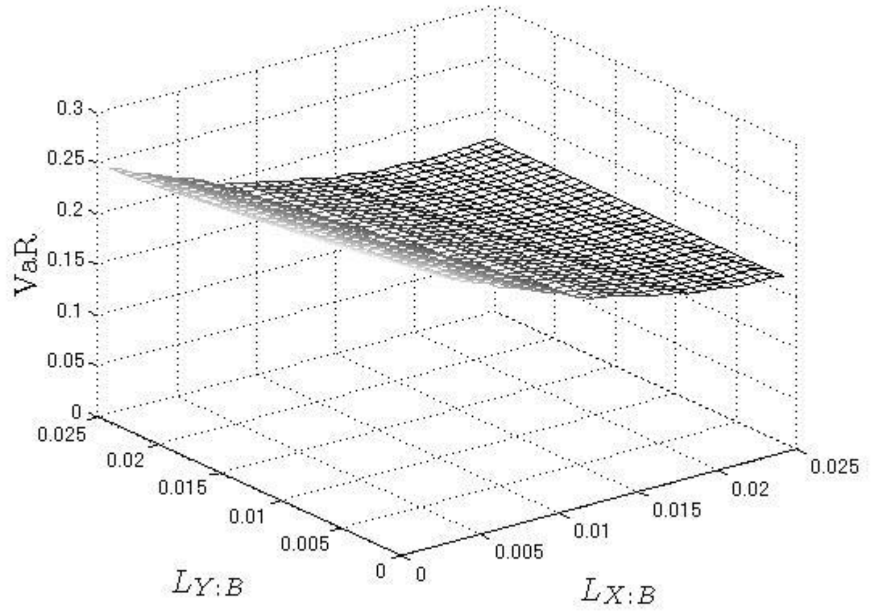

Figure 4.4. $VaR_{Y:Swap}(\bullet, L^{X:B}, \bullet, L^{Y:B}, p, t, \alpha)$

In order to confirm the effect of the swapping, we define

$$\Phi_X^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)$$
$$= \left(VaR_X(L^{X:A}, L^{X:B}, t, \alpha) - VaR_{X:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)\right)^+ \quad (4.3)$$

and

$$\Phi_Y^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha) \quad (4.4)$$
$$= \left(VaR_Y(L^{Y:A}, L^{Y:B}, t, \alpha) - VaR_{Y:Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)\right)^+$$

where $(a)^+ = \max\{a,0\}$ .Figures 4.5 and 4.6 exhibit $\Phi_X^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)$ and $\Phi_Y^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t,)$ as functions of $L^{X:B}$ and $L^{Y:B}$ respectively.

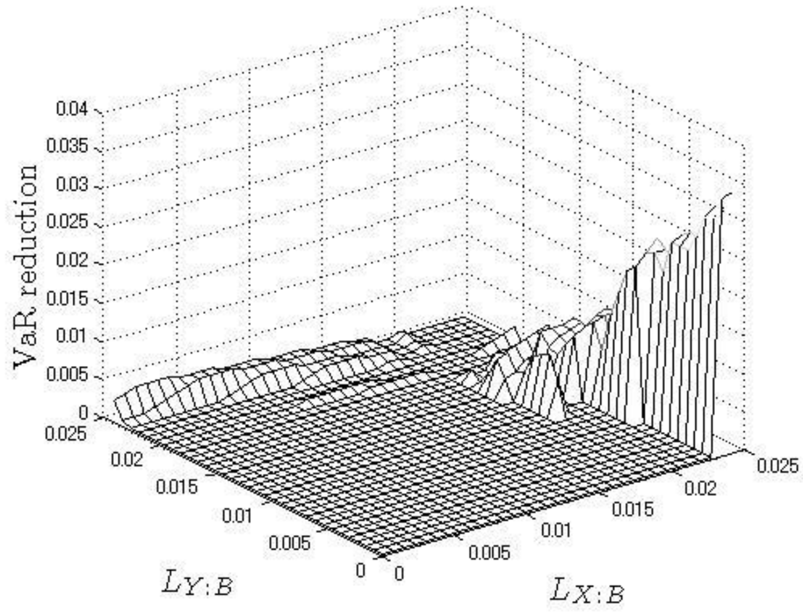

Figure 4.5. $\Phi_X^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t, \alpha)$

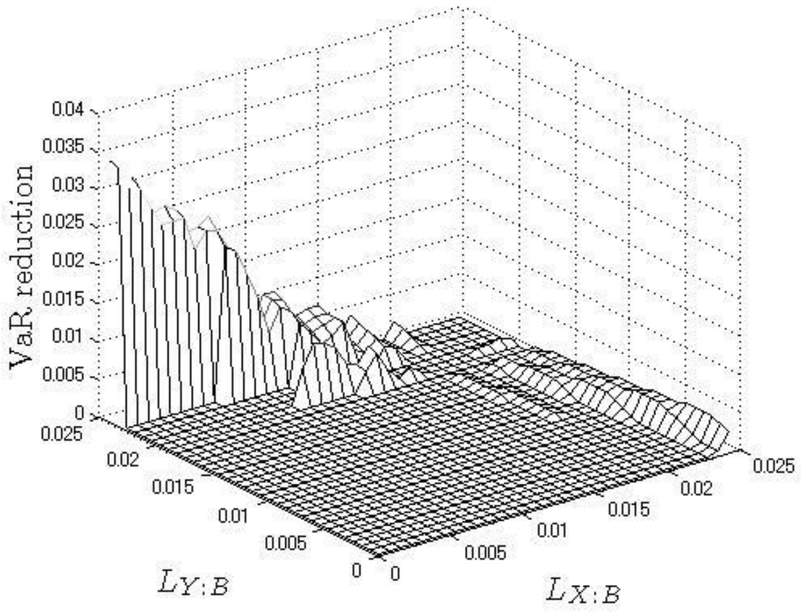

Figure 4.6. $\Phi_Y^{Swap}(L^{X:A}, L^{X:B}, L^{Y:A}, L^{Y:B}, p, t,)$

It is clear that the swapping could benefit both Bank *X* and Bank *Y* when either $L^{X:B}$ or $L^{Y:B}$ is large, but never when both are large. Therefore, swapping is effective when one of the two banks has a higher risk with the e-commerce corporations in *B* than the other. When the risk level of the two banks for the e-commerce corporations in *B* is similar, the swapping would not be effective.

5. CONCLUSION

The collapse of the IT industry in 2002, combined with the Lehman shock in 2008, slowed the capital flow into e-commerce significantly, so it is important to find a way to facilitate the capital flow into e-commerce by controlling the risks involved. In this paper, we developed and analyzed a mathematical model for two banks X and Y to swap credit risks across two industrial sectors: an industrial sector *A* that is not involved in e-commerce, and another industrial sector *B* that relies upon e-commerce. When Bank *X* and Bank *Y* provide loans to corporations in *A* and *B*, the paper shows that a swapping scheme could be devised between *X* and *Y* so as to improve the Value-at-Risk for both *X* and *Y*, thereby providing incentive for *X* and *Y* to invest incorporations that are involved in e-commerce.

Based on a Markov modulated Poisson process(MMPP) developed by Takada, Sumita & Takahashi [1], an efficient computational procedure were established to solve the Value-at-Risk problems of Bank *X* and Bank *Y*. Numerical examples demonstrated that the swapping could benefit both Bank *X* and Bank *Y* when one of the two banks has higher risk related to e-commerce corporations than the other, thereby providing incentive for the two banks to invest in e-commerce corporations. If the two banks' risk levels related to the e-commerce corporations is similar, the swapping may not be effective.

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